Optimal controller/observer gains of discounted-cost LQG systems

Hildo Bijl and Thomas B. Schön

***Abstract*—The linear-quadratic-Gaussian (LQG) control paradigm is well-known in literature. The strategy of minimizing the cost function is available, both for the case where the state is fully known and where it is estimated through an observer. The situation is different when the cost function has an exponential discount factor, also known as a prescribed degree of stability. In this case, the optimal control strategy is only available when the state is fully known. This paper builds onward from that result, deriving an optimal control strategy when working with an estimated state. Expressions for the resulting optimal expected cost are also given. The result is illustrated via an experimental validation.**

***Index Terms*—Linear systems, cost function, LQG, optimal control, Riccati equation.**

I. Introduction

CONSIDER the continuous-time linear system[1]

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + \boldsymbol{v}(t), \tag{1a}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t) + \boldsymbol{w}(t), \tag{1b}$$

with $\boldsymbol{x}$ the state, $\boldsymbol{u}$ the input, $\boldsymbol{y}$ the output, $\boldsymbol{v}$ and $\boldsymbol{w}$ Gaussian white noise with respective intensities V and W, and A, B, C and D the system matrices. We assume that the initial state $\boldsymbol{x_0}$ is unknown but distributed according to a Gaussian distribution with $\boldsymbol{\mu_0} = \mathbb{E}[\boldsymbol{x_0}]$ and $\Sigma_0 = \mathbb{E}\left[\boldsymbol{x_0}\boldsymbol{x_0}^T\right]$. Note that Σ_0 is *not* the variance of $\boldsymbol{x_0}$.

Our goal is to control system (1) such as to minimize the discounted (exponential) quadratic cost function

$$J(T) = \int_0^T e^{2\alpha t}\left(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)\right)\,dt, \tag{2}$$

with $J(T)$ the cost, α the discount exponent/prescribed degree of stability, and $Q \geq 0$ and $R > 0$ weight matrices. In particular, we will optimize the infinite-time expected cost $\mathbb{E}[J]$, with

$$J = \lim_{T\to\infty} J(T). \tag{3}$$

Our main contribution in this paper is that we derive the optimal controller and observer gains for the continuous-time linear system (1) such that the expected cost $\mathbb{E}[J]$ given in (3) is minimized.

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[1] From a formal point of view the system notation of (1) is incorrect, because $\boldsymbol{v}(t)$ and $\boldsymbol{w}(t)$ are not measurable with nonzero probability. However, since this notation is common in the control literature, we will stick with it. For methods to properly deal with stochastic differential equations, see [1].

We start by providing a brief literature overview (Section II) investigating the status of literature on this subject by providing links to related work on similar problems. We then examine a few already known Theorems in Section III and expand on this in Section IV, solving the stated problem. We verify the results with an experiment in Section V and state conclusions in Section VI.

II. Related work

Linear-Quadratic-Gaussian (LQG) systems—linear systems with a quadratic cost function subject to Gaussian noise—have been thoroughly investigated in the past. This was especially true near the 1960s, with for instance the publication of the Kalman filter [2], [3].

The discoveries from the decades afterwards have been summarized in numerous textbooks. Examples include the books by [4, Chapter 7], [5, Chapter 5], [6, Chapter 1], [7, Chapters 3, 8], [8, Chapter 6], [9, Chapter 10], [10, Chapter 9] and [11, Chapter 4]. All these books examine the non-discounted cost function (with $\alpha = 0$), but it is only [7] (Section 3.5) that also considers the discounted cost function, presenting results from an earlier paper [12]. Here it was shown that discounting the cost function is equivalent to prescribing a degree of stability.

The prescribed degree of stability is actually a relevant problem in that it is a generalization of the regular LQG paradigm with the non-discounted cost function. There is also a variety of applications of this idea, such as fault tolerant flight control [13], spacecraft guidance [14] and robot manipulators [15]. However, to the best of the authors knowledge there are still fundamental properties remaining to be established and our main contribution in this paper is to provide one of those. The work [12] only examined the situation where the state is assumed to be known. If the state can only be observed through noisy output measurements—a familiar problem for the non-discounted cost function—then we are not aware of any work that jointly optimize the controller and the state estimator. The closest is the work by [16], who strived to set up a state estimator with minimal mean squared error, given a prescribed convergence rate. However, that work ignored the uncertainty in the initial state and did not examine the problem of jointly optimizing the controller and observer gains. In fact, it was not mentioned whether the separation principle still holds or not when using the discounted cost function. Hence, the problem of jointly optimizing the controller and observer gains, subject to a discounted cost function, appears to be an open problem.

III. Background on known properties

Before we start with the actual problem, let us briefly examine the cases for which the optimal control law is already known. When doing so, we also aim to give the corresponding expected cost, using expressions from [17]. We begin with the situation where the state is fully known, first examining the non-discounted cost ($\alpha = 0$) and moving on to the discounted cost ($\alpha \neq 0$). Then we add an observer and do the same.

A. The non-discounted case with fully known state

We examine the non-discounted case ($\alpha = 0$) where the state $\boldsymbol{x}(t)$ is fully known (for instance when $C = I$ and $W = 0$). The solution of this problem is well-known.

Theorem 1. *Consider system* (1)*, where the state is assumed known. If* (A, B) *is stabilizable, then the optimal control law minimizing the expected non-discounted cost* $\mathbb{E}[J]$ *(i.e., with* $\alpha = 0$*) is a linear control law* $\boldsymbol{u}(t) = -F\boldsymbol{x}(t)$*, where*

$$F = R^{-1}B^TX, \tag{4}$$

and X *is the solution to the Riccati equation*

$$A^TX + XA + Q - XBR^{-1}B^TX = 0. \tag{5}$$

When $V = 0$*, the corresponding expected cost equals*

$$\mathbb{E}[J] = \mathbb{E}\left[\boldsymbol{x_0}^T X \boldsymbol{x_0}\right] = tr\left(X\Sigma_0\right). \tag{6}$$

When $V \neq 0$*, then* $\mathbb{E}[J(T)] \to \infty$*, but the steady-state cost rate equals*

$$\lim_{T\to\infty} \frac{d\mathbb{E}[J(T)]}{dT} = tr\left(XV\right). \tag{7}$$

Proof. See any of the aforementioned books; for example [5, Theorem 3.9]. □

It is important to note that the noise V does not affect the optimal control law whatsoever. The control strategy can be summarized as 'Bring the state back to zero as efficiently as possible, from any disturbance that may occur.' Which exact disturbance occurs is irrelevant here.

B. The discounted case with fully known state

Let us now add the discount exponent α. Note that α can be both positive (a prescribed degree of stability) and negative (a discount exponent), but for ease of writing we always call it a discount exponent. We also define $A_\alpha = A + \alpha I$.

Theorem 2. *Consider system* (1)*, where the state is assumed known. If* (A_α, B) *is stabilizable, then the optimal control law minimizing the expected discounted cost* $\mathbb{E}[J]$ *is a linear control law* $\boldsymbol{u}(t) = -F_\alpha\boldsymbol{x}(t)$*, where*

$$F_\alpha = R^{-1}B^TX_\alpha, \tag{8}$$

and X_α *is the solution to the Riccati equation*

$$A_\alpha^TX_\alpha + X_\alpha A_\alpha + Q - X_\alpha BR^{-1}B^TX_\alpha = 0. \tag{9}$$

The corresponding expected cost (for both zero and nonzero V*) when* $\alpha < 0$ *equals*

$$\mathbb{E}[J] = tr\left(\left(\frac{-X_\alpha}{2\alpha}\right)(V - 2\alpha\Sigma_0)\right). \tag{10}$$

When $\alpha \geq 0$*, then* $\mathbb{E}[J(T)] \to \infty$ *as* $T \to \infty$*.*

Proof. The key insight [7, Section 3.5] is to define

$$\tilde{\boldsymbol{x}}(t) = e^{\alpha t}\boldsymbol{x}(t), \quad \tilde{\boldsymbol{u}}(t) = e^{\alpha t}\boldsymbol{u}(t), \quad \tilde{\boldsymbol{v}}(t) = e^{\alpha t}\boldsymbol{v}(t). \tag{11}$$

Note that $\tilde{\boldsymbol{v}}(t)$ is Gaussian white noise with a time-varying intensity $\tilde{V}(t) = e^{2\alpha t}V$. The time-derivative of $\tilde{\boldsymbol{x}}(t)$ now follows as

$$\dot{\tilde{\boldsymbol{x}}}(t) = A_\alpha\tilde{\boldsymbol{x}}(t) + B\tilde{\boldsymbol{u}}(t) + \tilde{\boldsymbol{v}}(t), \tag{12}$$

where $\tilde{\boldsymbol{x}}(0) = \boldsymbol{x}(0)$. At the same time, expression (2) for the cost $J(T)$ turns into

$$J(T) = \int_0^T \left(\tilde{\boldsymbol{x}}^T(t)Q\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{u}}^T(t)R\tilde{\boldsymbol{u}}(t)\right) dt. \tag{13}$$

We can directly apply Theorem 1 to this situation, proving the theorem except for (10). This expression follows directly from [17, Theorem 2]. □

It is important to note what 'optimality' means here. After all, we have process noise continuously exciting the system. If $\alpha > 0$, then the infinite-time cost J will be infinite, regardless of the control law used. A control law is now optimal when the corresponding expected cost $\mathbb{E}[J(T)]$ is smaller than the expected cost $\mathbb{E}[J'(T)]$ corresponding to any other control law, in the limit $T \to \infty$. Note that this does not imply that $\mathbb{E}[J(T)] \leq \mathbb{E}[J'(T)]$ for all times T, but only for sufficiently large T.

An interesting property of the above theorem is that the control law ensures that all the closed-loop system eigenvalues are smaller than $-\alpha$ [7, Section 3.5]. This is why α (when positive) is known as the prescribed degree of stability. We do not directly place the eigenvalues like [18], but we do prescribe them to satisfy $\lambda_i < -\alpha$.

Generally, positive values of α result in more aggressive controllers (higher gains) while negative values of α result in more lazy controllers (lower gains). Intuitively speaking, when α is negative, the controller does not bother making a large effort to 'fix' future states, because these future states will hardly matter anyway.

C. The non-discounted case with unknown state

We now consider the complete system (1), where the state is no longer assumed to be known. We do for now assume that $\alpha = 0$. The common solution in dealing with the unknown state is to set up an observer with state estimate $\hat{\boldsymbol{x}}$, which is updated according to

$$\dot{\hat{\boldsymbol{x}}}(t) = A\hat{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + K\left(\boldsymbol{y}(t) - C\hat{\boldsymbol{x}}(t) - D\boldsymbol{u}(t)\right), \tag{14}$$

subject to some initial state estimate $\hat{\boldsymbol{x}}(0)$. We also define the state estimation error as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. To minimize this error (i.e., its variance) we can use the following Theorem.

Theorem 3. *Consider system* (1)*. If* (A, C) *is detectable, then the optimal observer gain minimizing the steady-state error covariance is*

$$K = EC^TW^{-1}, \tag{15}$$

where E is the optimal steady-state error covariance, found through

$$AE + EA^T + V - EC^T W^{-1} CE = 0. \tag{16}$$

Proof. This is the famous Kalman-Bucy filter from [3]. A proof can also be found in [11, Section 4.3]. □

Theorem 4. *Consider system* (1)*. If (A, B) is stabilizable and (A, C) is detectable, then the optimal control law minimizing the expected non-discounted cost (i.e., with $\alpha = 0$) is a linear control law $\boldsymbol{u}(t) = -F\hat{\boldsymbol{x}}(t)$, with F given by* (4)*, $\hat{\boldsymbol{x}}(t)$ following from* (14) *and the observer gain K taken as* (15)*. The resulting expected steady-state cost rate is given by*

$$\begin{aligned}\lim_{T\to\infty} \frac{d\mathbb{E}[J(T)]}{dT} &= tr\left(XKWK^T\right) + tr\left(EQ\right) \\ &= tr\left(XV\right) + tr\left(EF^T RF\right), \end{aligned} \tag{17}$$

with X the solution of (5) *and E the solution of* (16)*.*

Proof. The optimal controller and observer gains follow from the separation principle. See for instance [11, Section 4.3]. This leaves us only with the proof for (17), which is given in the Appendix, as Theorem 6. □

IV. Optimizing the discounted cost function

In this section we derive the main result: the optimal controller/observer gains for the complete system. The following Theorem shows that the separation principle still holds, albeit in an adjusted form, when using the discounted cost function.

Theorem 5. *Consider system* (1)*. If (A_α, B) is stabilizable and (A_α, C) is detectable, then the optimal control law minimizing the expected discounted cost $\mathbb{E}[J]$ is a linear control law $\boldsymbol{u}(t) = -F_\alpha \hat{\boldsymbol{x}}(t)$, with F_α given by* (8) *and X_α given by* (9)*. Identically to* (14)*, $\hat{\boldsymbol{x}}(t)$ is provided by the observer*

$$\dot{\hat{\boldsymbol{x}}}(t) = A\hat{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + K_\alpha\left(\boldsymbol{y}(t) - C\hat{\boldsymbol{x}}(t) - D\boldsymbol{u}(t)\right), \tag{18}$$

where $\hat{\boldsymbol{x}}_{\boldsymbol{0}}$ is set to $\boldsymbol{\mu}_{\boldsymbol{0}}$, the observer gain K_α is given by

$$K_\alpha = E_\alpha C^T W^{-1} \tag{19}$$

and E_α is the solution to the Riccati equation

$$\begin{aligned} A_\alpha E_\alpha + E_\alpha A_\alpha^T + \left(V - 2\alpha\left(\Sigma_0 - \boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T\right)\right) \\ - E_\alpha C^T W^{-1} C E_\alpha = 0. \end{aligned} \tag{20}$$

The corresponding expected cost for $\alpha < 0$ equals

$$\begin{aligned}\mathbb{E}\left[J\right] &= tr\left(\left(\frac{-X_\alpha}{2\alpha}\right)\left(K_\alpha W K_\alpha^T - 2\alpha\boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T\right) + \left(\frac{-E_\alpha}{2\alpha}\right)Q\right) \\ &= tr\left(\left(\frac{-X_\alpha}{2\alpha}\right)\left(V - 2\alpha\Sigma_0\right) + \left(\frac{-E_\alpha}{2\alpha}\right)F_\alpha^T R F_\alpha\right). \end{aligned} \tag{21}$$

When $\alpha \geq 0$, then $\mathbb{E}[J(T)] \to \infty$ as $T \to \infty$.

Proof. This follows from Theorem 4. To see how, we need to carefully note the differences between the two problems. Our system, with the control law and the observer, can be written in the form

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\boldsymbol{x}}}(t) \\ \dot{\boldsymbol{e}}(t)\end{bmatrix} &= \begin{bmatrix} A - BF & -KC \\ 0 & A - KC\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}}(t) \\ \boldsymbol{e}(t)\end{bmatrix} + \begin{bmatrix} K\boldsymbol{w}(t) \\ K\boldsymbol{w}(t) + \boldsymbol{v}(t)\end{bmatrix} \\ &= \tilde{A}\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{v}}(t). \end{aligned} \tag{22}$$

This notation was also used in proving Theorem 4 in the Appendix. Now, since $\hat{\boldsymbol{x}}_{\boldsymbol{0}} = \boldsymbol{\mu}_{\boldsymbol{0}}$, the initial state satisfies

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{0}} = \mathbb{E}\left[\tilde{\boldsymbol{x}}_{\boldsymbol{0}}\right] = \mathbb{E}\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{0}} \\ \boldsymbol{e}_{\boldsymbol{0}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\mu}_{\boldsymbol{0}} \\ \boldsymbol{0}\end{bmatrix}, \tag{23a}$$

$$\tilde{\Sigma}_0 = \mathbb{E}\left[\tilde{\boldsymbol{x}}_{\boldsymbol{0}}\tilde{\boldsymbol{x}}_{\boldsymbol{0}}^T\right] = \begin{bmatrix}\boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T & 0 \\ 0 & \Sigma_0 - \boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T\end{bmatrix}. \tag{23b}$$

According to Theorem 2, the expected cost equals

$$\mathbb{E}\left[J\right] = \text{tr}\left(\left(\frac{-\tilde{X}_\alpha}{2\alpha}\right)\left(\tilde{V} - 2\alpha\tilde{\Sigma}_0\right)\right). \tag{24}$$

This is the quantity that we need to optimize, contrary to Theorem 4 where we had to optimize tr $\left(\tilde{X}\tilde{V}\right)$. To turn one situation into the other, we hence need to substitute $\tilde{A}$ by $\tilde{A}_\alpha$, which is equivalent to replacing A by A_α. Similarly, we must substitute $\tilde{V}$ for $\left(\tilde{\Sigma}_0 - \frac{\tilde{V}}{2\alpha}\right)$, which is equivalent to replacing W by $\frac{-W}{2\alpha}$ and V by $\left(\Sigma_0 - \boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T - \frac{V}{2\alpha}\right)$. In addition, due to the top left term of $\tilde{\Sigma}_0$, expression (31a) also gets an extra term $\boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T$. After these replacements and adjustments, the problems are indeed identical. Elementary algebra subsequently turns Theorem 4 into the desired result. □

It is interesting to note that, for the discounted cost, the optimal controller gains stay the same as compared to Theorem 2, and Theorem 2 was just a minor adjustment with respect to Theorem 1. The controller still trades off aggressively controlling the state (for high Q) with being frugal at applying input (for high R).

The situation is different for the observer. In the non-discounted case (Theorem 4) it trades off aggressive adjustments of $\hat{\boldsymbol{x}}$ to compensate for process noise (for high V) with cautious adjustments due to the presence of measurement noise (for high W). In the discounted case, also the variance $\Sigma_0 - \boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T$ of the initial state matters. If the initial state is highly uncertain (large variance) and if the present matters more than the future (large negative α) then we get a more aggressively adjusting state estimator. Roughly put, the state estimator wants to 'fix' the uncertainty in the state while things still matter. This is contrary to the default effect of α. In the absence of uncertainty (when $\Sigma_0 - \boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T = 0$) a large negative value of α results in a lazy observer, just like it gives a lazy controller.

V. Experimental verification

To check the derived equations, we set up a numerical experiment using Matlab (R2015a). Ideally, this experiment is as basic as possible. Applying the derived equations to an industrial experiment is hardly different from applying the well-known Theorem 4 (regular LQG control) to the same experiment, so doing that will not provide any new insights.

We set up an experiment with the following parameters,

$$\boldsymbol{\mu}_{\boldsymbol{0}} = \begin{bmatrix}10 \\ -8\end{bmatrix}, \qquad \Sigma_0 = \begin{bmatrix}1 & 0.5 \\ 0.5 & 2\end{bmatrix} + \boldsymbol{\mu}_{\boldsymbol{0}}\boldsymbol{\mu}_{\boldsymbol{0}}^T, \tag{25a}$$

$$A = \begin{bmatrix}2 & -1 \\ 0.5 & -3\end{bmatrix}, \qquad B = \begin{bmatrix}0.5 \\ -1\end{bmatrix}, \tag{25b}$$

$$C = \begin{bmatrix}0 & 1\end{bmatrix}, \qquad D = \begin{bmatrix}0\end{bmatrix}. \tag{25c}$$

In addition, we use $V = 0.3I$, $W = 0.01$, $Q = I$, $R = 0.2$ and $\alpha = -0.1$. The system matrices have been mostly chosen randomly, but the other parameters are selected such that all factors (the mean initial state, the initial state variance, the process noise and the measurement noise) contribute roughly equally to the final cost.

For the given set-up, we can calculate the optimal controller/observer gains. These turn out to be

$$F_\alpha = \begin{bmatrix} 6.92 & -1.41 \end{bmatrix}, \qquad K_\alpha = \begin{bmatrix} 80.2 \\ 9.54 \end{bmatrix}. \tag{26}$$

The resulting expected cost, according to (21), becomes $\mathbb{E}[J] = 1\,093$. We can verify this through numerical simulations. After running 10^5 simulations, each time initializing the system in a random initial state and applying the appropriate process/measurement noise for $T = 40$ seconds, while keeping track of the cost J, the mean cost of all simulations was $1\,094$. The tiny difference from the predicted mean can partly be explained by statistical deviations, and partly by minor inaccuracies in the discretization of the simulation. However, the result is still in close agreement with the theory provided by Theorem 5.

We should also check whether the given controller settings indeed minimize the discounted cost. To do so, we can make minor adjustments to the gains F_α and K_α and investigate the resulting change in expected cost. The results are shown in Table I. Here we see that small adjustments to either of the gains, both positive and negative, will only increase the expected cost $\mathbb{E}[J]$. This confirms that the given controller parameters are indeed optimal.

TABLE I
THE EFFECTS OF MINOR ADJUSTMENTS OF ELEMENTS OF THE OPTIMAL CONTROLLER/OBSERVER GAINS F_α AND K_α ON THE EXPECTED COST $\mathbb{E}[J]$. NOTE THAT THE CHANGE IN COST IS ALWAYS POSITIVE: THE COST ALWAYS INCREASES. NUMBERS WERE DERIVED USING (21).

Change in F_α	**Effect on $\mathbb{E}[J]$**	**Change in K_α**	**Effect on $\mathbb{E}[J]$**
$\Delta F_{\alpha,1} = -0.1$	$\Delta\mathbb{E}[J] = 0.23$	$\Delta K_{\alpha,1} = -0.5$	$\Delta\mathbb{E}[J] = 0.059$
$\Delta F_{\alpha,1} = 0.1$	$\Delta\mathbb{E}[J] = 0.22$	$\Delta K_{\alpha,1} = 0.5$	$\Delta\mathbb{E}[J] = 0.058$
$\Delta F_{\alpha,2} = -0.1$	$\Delta\mathbb{E}[J] = 0.41$	$\Delta K_{\alpha,2} = -0.1$	$\Delta\mathbb{E}[J] = 0.087$
$\Delta F_{\alpha,2} = 0.1$	$\Delta\mathbb{E}[J] = 0.41$	$\Delta K_{\alpha,2} = 0.1$	$\Delta\mathbb{E}[J] = 0.088$

VI. CONCLUSIONS AND RECOMMENDATIONS

It is now possible to find the optimal controller and observer gains of an LQG system with discounted cost using the expressions provided in Theorem 5. The explicit expressions for the expected cost are also provided. Finally, we hope that this paper also serves the purpose of providing a focused overview when it comes to this part of control engineering.

Future work on this subject can look into replacing the discount exponent α by a discount matrix. This seems to be a mostly straightforward problem with fascinating additional tuning possibilities. A more complicated matter would be to figure out how a finite time window T affects the optimal controller/observer parameters for various α. Yet another avenue for further research would be to look into time-varying systems, similarly to [16], and investigate whether the same results are still applicable or not.

APPENDIX

PROOF OF THE NON-DISCOUNTED COST EXPRESSION

Theorem 6. *Expression* (17) *from Theorem 4 holds, subject to the conditions given.*

Proof. To prove this, we first note that our system, with the control law and the observer, can be written in a slightly adjusted notation as

$$\begin{bmatrix} \dot{\hat{\boldsymbol{x}}} \\ \dot{\boldsymbol{e}} \end{bmatrix} = \begin{bmatrix} A - BF & -KC \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}} \\ \boldsymbol{e} \end{bmatrix} + \begin{bmatrix} K\boldsymbol{w} \\ K\boldsymbol{w} + \boldsymbol{v} \end{bmatrix}, \tag{27}$$

where we have omitted the dependency on time (t) for brevity. If we (re-)define

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \hat{\boldsymbol{x}} \\ \boldsymbol{e} \end{bmatrix}, \tag{28a}$$

$$\tilde{A} = \begin{bmatrix} A - BF & -KC \\ 0 & A - KC \end{bmatrix}, \tag{28b}$$

$$\tilde{V} = \begin{bmatrix} KWK^T & KWK^T \\ KWK^T & KWK^T + V \end{bmatrix}, \tag{28c}$$

$$\tilde{Q} = \begin{bmatrix} Q + F^T RF & -Q \\ -Q & Q \end{bmatrix}, \tag{28d}$$

then we can apply [17, Theorem 3] to find the expected steady-state cost rate. This becomes, also through [17, Theorem 16],

$$\lim_{T\to\infty} \frac{d\mathbb{E}[J(T)]}{dT} = \operatorname{tr}\left(X_{\tilde{A}}^{\tilde{V}} \tilde{Q}\right) = \operatorname{tr}\left(\tilde{V} \bar{X}_{\tilde{A}}^{\tilde{Q}}\right), \tag{29}$$

where we use the notation from [17] for $X_{\tilde{A}}^{\tilde{V}}$ and $\bar{X}_{\tilde{A}}^{\tilde{Q}}$. That is, these parameters per definition are the solutions of the Lyapunov equations

$$\tilde{A} X_{\tilde{A}}^{\tilde{V}} + X_{\tilde{A}}^{\tilde{V}} \tilde{A}^T + \tilde{V} = 0, \tag{30a}$$

$$\tilde{A}^T \bar{X}_{\tilde{A}}^{\tilde{Q}} + \bar{X}_{\tilde{A}}^{\tilde{Q}} \tilde{A} + \tilde{Q} = 0. \tag{30b}$$

We will initially use the first of these two expression, expanding (30a) into

$$(A - BF)\, X_{\tilde{A},11}^{\tilde{V}} - KC X_{\tilde{A},12}^{\tilde{V}} + X_{\tilde{A},11}^{\tilde{V}} (A - BF)^T - X_{\tilde{A},12}^{\tilde{V}} C^T K^T + KWK^T = 0, \tag{31a}$$

$$(A - BF)\, X_{\tilde{A},12}^{\tilde{V}} - KC X_{\tilde{A},22}^{\tilde{V}} + X_{\tilde{A},12}^{\tilde{V}} (A - KC)^T + KWK^T = 0, \tag{31b}$$

$$(A - KC)\, X_{\tilde{A},22}^{\tilde{V}} + X_{\tilde{A},22}^{\tilde{V}} (A - KC)^T + KWK^T + V = 0. \tag{31c}$$

For the given values of F and K, we can directly find[2] that $X_{\tilde{A},12}^{\tilde{V}} = 0$. Subsequently, the solution for (31a) equals $X_{\tilde{A},11}^{\tilde{V}} = X$ (since (31a) then equals (5)) and the solution

[2]This actually also follows from the separation principle. We can note that $X_{\tilde{A},12}^{\tilde{V}} = \lim_{t\to\infty} \mathbb{E}\left[\hat{\boldsymbol{x}}(t)\boldsymbol{e}^T(t)\right]$, and the separation principle implies that this quantity becomes zero; see [7, Section 8.2].

for (31c) equals $X^{\tilde{V}}_{\tilde{A},22} = E$ (since (31c) then equals (16)). It immediately follows that

$$\lim_{T\to\infty} \frac{d\mathbb{E}[J(T)]}{dT} = \text{tr}\left(X^{\tilde{V}}_{\tilde{A}}\tilde{Q}\right) \\ = \text{tr}\left(X\big(Q + F^T RF\big) + EQ\right). \qquad (32)$$

Through [17, Theorem 16], we directly find the first half of expression (17). To find the second half, we need to apply the above methodology in an identical way to (30b) instead of (30a). When doing so, we complete our proof. □

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